

Persistence of Random Walk Records

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We study records generated by Brownian particles in one dimension. Specifically, we investigate an ordinary random walk and define the record as the maximal position of the walk. We compare the record of an individual random walk with the mean record, obtained as an average over infinitely many realizations. We term the walk “superior” if the record is always above average, and conversely, the walk is said to be “inferior” if the record is always below average. We find that the fraction of superior walks, S , decays algebraically with time, $S \sim t^{-\beta}$, in the limit $t \rightarrow \infty$, and that the persistence exponent is nontrivial, $\beta = 0.382258 \dots$. The fraction of inferior walks, I , also decays as a power law, $I \sim t^{-\alpha}$, but the persistence exponent is smaller, $\alpha = 0.241608 \dots$. Both exponents are roots of transcendental equations involving the parabolic cylinder function. To obtain these theoretical results, we analyze the joint density of superior walks with given record and position, while for inferior walks it suffices to study the density as function of position.

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I. INTRODUCTION

The record, defined as the extremum in a sequence of variables, is a useful characteristic of a dataset. Extreme value theory [1–5] and analysis of sequences of uncorrelated random variables [6, 7] provide the basis for understanding record statistics. Records in problems ranging from finance [8–11] and sport [12] to random structures [13, 14] and complex networks [15–17] typically involve sequences of *correlated* random variables. However, current theoretical understanding of extreme values of correlated random variables is still far from complete [18–20].

Brownian trajectories are prime examples of correlated time series [21–23]. Established record statistics of discrete time one-dimensional random walks include the distribution of the number of records and the mean duration of the longest record [24, 25]. Records in ensembles of random walks, especially the distribution of the maximum, have also been studied both for independent [26, 27] and for interacting random walks [28, 29]. However, much less is known about random walk records in higher dimensions [30, 31].

Recent studies show that first-passage [32] and persistence properties [33, 34] of records have rich phenomenology [35, 36]. For a sequence of uncorrelated random variables, the probability that all records are above average decays algebraically with sequence length and this behavior is governed by a nontrivial persistence exponent. Such persistence characteristics were used to analyze earthquake data [35, 36]. In this article, we study similar persistence characteristics of random walk records.

We consider a discrete time random walk in one dimension. The walk starts at the origin, $x(0) = 0$, and in each time step the walk makes a jump: $x(t+1) = x(t) + \Delta_t$. The jump lengths Δ_t are independent random variables chosen from a symmetric distribution with finite variance: $\langle \Delta \rangle = 0$ and $\langle \Delta_i \Delta_j \rangle = \delta_{ij} \langle \Delta^2 \rangle$ with $\langle \Delta^2 \rangle < \infty$.

The record $r(t)$ is defined as the maximal position of

the random walk in the time interval $(0, t)$

$$r(t) = \max\{x(0), x(1), x(2), \dots, x(t)\}. \quad (1)$$

We compare the record with the average record $a(t) = \langle r(t) \rangle$ where the brackets denote an average over all possible realizations of the random process governing the position $x(t)$. Specifically, we compare the sequence of records $\{r(0), r(1), r(2), \dots, r(t)\}$ generated by the random walk with the sequence of average records $\{a(0), a(1), a(2), \dots, a(t)\}$. We call a random walk *superior* if all records exceed the average, $r(\tau) \geq a(\tau)$ for all $\tau = 0, 1, 2, \dots, t$. Similarly, we define an *inferior* walk as one for which all records trail the average, $r(\tau) \leq a(\tau)$ for all $\tau = 0, 1, 2, \dots, t$.

We now define $S(t)$ and $I(t)$ as the probability that at time t , a walk is superior or inferior. Our main result is that the probabilities $S(t)$ and $I(t)$ decay algebraically with time

$$S \sim t^{-\beta} \quad \text{and} \quad I \sim t^{-\alpha} \quad (2)$$

as $t \rightarrow \infty$. The persistence exponents are transcendental numbers $\beta = 0.382258 \dots$ and $\alpha = 0.241608 \dots$. Both exponents are related to roots of the parabolic cylinder function,

$$D_{2\beta+1}(\sqrt{2/\pi}) = 0 \quad \text{and} \quad D_{2\alpha}(-\sqrt{2/\pi}) = 0. \quad (3)$$

The asymptotic behaviors (2)–(3) apply as long as the jump length distribution has zero mean and finite variance. Hence, we can restrict our attention to random walks with unit jump length: $\Delta = 1$ and $\Delta = -1$ are chosen with equal probabilities.

The rest of this paper is organized as follows. In Section II, we briefly summarize basic properties of the record including the distribution of records and the mean record. Since record is coupled to position, we study the joint density of superior walks with given record and position. This distribution obeys the diffusion equation,

and we obtain the long time asymptotic behavior using scaling analysis (Section III). In the complementary case of inferior walks the analysis simplifies because it suffices to consider only the position (Section IV). A few generalizations are mentioned in Section V, and concluding remarks are given in Section VI.

II. THE AVERAGE RECORD

We use a simple random walk as a model for Brownian motion in one dimension [23, 37]. The random walk starts at the origin, $x = 0$ at time $t = 0$, and in each time step, its position changes by a fixed amount

$$x(t+1) = \begin{cases} x(t) - 1 & \text{with prob. } 1/2; \\ x(t) + 1 & \text{with prob. } 1/2. \end{cases} \quad (4)$$

With these jump rules, the average position does not change, $\langle x(t) \rangle = 0$, while the mean square displacement equals time $\langle x^2(t) \rangle = t$.

The record, defined in Eq. (1), equals the maximum position to date. For a simple random walk, the average record grows as the square root of time

$$a(t) \simeq A\sqrt{t}, \quad \text{with } A = \sqrt{2/\pi}. \quad (5)$$

This behavior represents the leading asymptotic behavior. Similar behavior holds as long as the jump length distribution has zero mean and a finite variance, and in general, the ratio between the average record and the mean square displacement approaches a constant, $a(t)/\sqrt{\langle x^2(t) \rangle} \rightarrow A$ in the limit $t \rightarrow \infty$ [21, 22].

Let $q(r, t)$ be the probability distribution function that the record equals r at time t . This quantity follows from the probability that $r(\tau) < n$ for all $\tau = 0, 1, 2, \dots, t$. The probability of this event is the same as the survival probability $Q(n, t)$ that a random walker starting at the origin never crosses n during the time interval $(0, t)$. The quantity $Q(n, t)$ is well known and can be conveniently expressed using the error function, $Q(n, t) = \text{erf}(n/\sqrt{2t})$ in the long-time limit [32, 37]. The probability that $r(t) = n$ is then $Q(n+1, t) - Q(n, t)$ which is asymptotically equivalent to $dQ(n, t)/dn$. As a result, the probability distribution function $q(r, t)$ is a one-sided Gaussian,

$$q(r, t) \simeq \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{r^2}{2t}\right) \quad (6)$$

for $r \geq 0$. In particular, we recover the probability that the random walk remains in negative half-space, $q(0, t) \simeq A/\sqrt{t}$ [32].

Let $M_n = \int_0^\infty dr r^n q(r, t)$ be the n th moment of the record distribution. The zeroth moment, $M_0 = 1$, reflects that the probability distribution is normalized, and the first moment, $M_1 \simeq A\sqrt{t}$, gives the average quoted in (5). The second moment $M_2 \simeq t$ gives the variance,

$$\langle r^2 \rangle - \langle r \rangle^2 = \left(1 - \frac{2}{\pi}\right)t. \quad (7)$$

The average (5), the variance (7), and the distribution function (6) show that the record grows as square-root of time, $r \sim \sqrt{t}$. Hence, the typical record mimics the behavior of the typical position $x \sim \sqrt{t}$.

III. SUPERIOR WALKS

We now focus on superior walks, that is, walks for which the record exceeds the average record, $r(\tau) \geq a(\tau)$ at all times $\tau \leq t$. Since record r is coupled to position x , we have to consider how the pair of coordinates (x, r) evolves with time. The position changes at each time step. However, the record may or may not change, and there are two possibilities. When $x < r$ the position changes but the record stays the same (see Fig. 1)

$$(x, r) \rightarrow \begin{cases} (x-1, r) & \text{with prob. } 1/2; \\ (x+1, r) & \text{with prob. } 1/2. \end{cases} \quad (8)$$

When $x = r$, the position changes and depending on jump direction, the record may increase,

$$(r, r) \rightarrow \begin{cases} (r-1, r) & \text{with prob. } 1/2; \\ (r+1, r+1) & \text{with prob. } 1/2. \end{cases} \quad (9)$$

As illustrated in Fig. 1, the position performs ordinary random walk in the $x-r$ plane, and there is also upward “slip” along the diagonal $x = r$.

Let $P(x, r, t)$ be the density of superior walks with position x and record $r \geq x$ at time t . A sum over all records and positions gives the probability that a random walk is superior

$$S(t) = \sum_{r \geq 0} \sum_{x \leq r} P(x, r, t). \quad (10)$$

In the discrete time formulation the sums are actually finite, $0 \leq r \leq t$ and $-t \leq x \leq r$.

The jump rules (8)–(9) imply the following recurrence equations which relate the density at time $t+1$ to the density at time t ,

$$P(x, r, t+1) = \frac{P(x-1, r, t) + P(x+1, r, t)}{2} \quad (11a)$$

$$P(r, r, t+1) = \frac{P(r-1, r, t) + P(r-1, r-1, t)}{2}. \quad (11b)$$

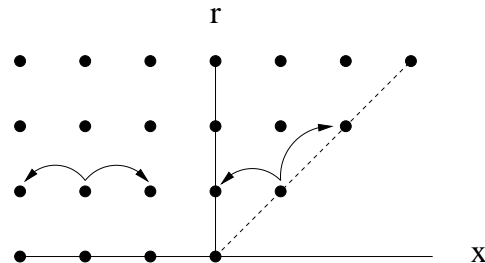


FIG. 1: Illustration of the jump processes (8) and (9).

Equation (11a) is valid for all $x < r$ and it corresponds to cases where the record was set before the final step. Equation (11b) describes the evolution of the density along the diagonal $x = r$ and it contains a contribution from walks in which the record was set at the final step.

For the random walk (4), position x and record r are discrete variables. Since we are interested in the long-time asymptotic behavior, we may treat these variables as continuous. The density $P(x, r, t)$ satisfies the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 P}{\partial x^2} \quad (12)$$

in the domain $-\infty < x < r$ and $r > 0$. This equation follows from the recurrence equation (11a), and it reflects that the position undergoes an ordinary diffusion process. To obtain (12), we replace the left-hand side in (11a) with a first-order Taylor expansion in time, $P + \partial P/\partial t$, and similarly, replace the right-hand-side with a second order expansion in position x . The diffusion equation (12) is subject to the boundary condition

$$2 \frac{\partial P}{\partial x} + \frac{\partial P}{\partial r} = 0 \quad (13)$$

on the diagonal $x = r$. This relation, which properly accounts for the upward slip along the boundary, can be derived from the recurrence equation (11b) by repeating the steps leading to (12).

We now introduce a new variable y which is a linear combination of record and position

$$y = 2r - x. \quad (14)$$

With this transformation of variables $(r, x) \rightarrow (r, y)$, the diffusion process takes place in the domain $y \geq r \geq 0$, and importantly, the boundary condition (13) simplifies to $\partial P/\partial r = 0$, along the diagonal $y = r$. According to equation (12), the density $P \equiv P(y, r, t)$ still obeys the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \quad (15)$$

in the domain $y > r \geq 0$.

As discussed in Section II, the position and the record both grow as square root of time, $x \sim \sqrt{t}$ and $r \sim \sqrt{t}$, and consequently, $y \sim \sqrt{t}$. Hence, the density of superior walks has the scaling form

$$P(y, r, t) \sim t^{-\beta-1} \Phi\left(\frac{y}{\sqrt{t}}, \frac{r}{\sqrt{t}}\right). \quad (16)$$

This scaling form is compatible with (10) and the algebraic decay $S \sim t^{-\beta}$. The scaling function $\Phi \equiv \Phi(Y, R)$ depends on the variables $Y = y/\sqrt{t}$ and $R = r/\sqrt{t}$ corresponding to the scaled position and record, respectively. For superior walks we have $y \geq r \geq a$ and from equation (5), we conclude $Y > R > A$ with $A = \sqrt{2/\pi}$. Hence,

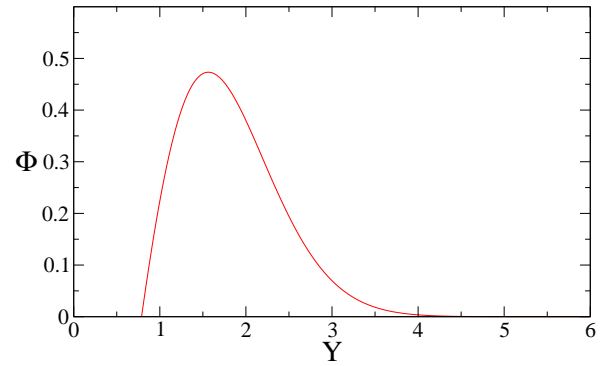


FIG. 2: The scaling function $\Phi(Y)$ versus the scaling variable Y . The scaling function vanishes at $Y = A = \sqrt{2/\pi}$.

we have the boundary condition $\Phi(Y, R) = 0$ on the line $R = A$.

The boundary condition $\partial P/\partial r = 0$ on the diagonal $y = r$ implies $\partial \Phi/\partial R = 0$ when $Y = R$. This suggests to seek a scaling function that depends on the variable Y alone, $\Phi \equiv \Phi(Y)$. By substituting the scaling form (16) into the diffusion equation (15), we find that Φ obeys the second-order ordinary differential equation

$$\Phi'' + Y\Phi' + 2(\beta + 1)\Phi = 0, \quad (17)$$

where prime denotes differentiation with respect to Y . The boundary condition is $\Phi(A) = 0$. The first two terms in (17) imply that Φ has a Gaussian tail, $\Phi \sim \exp(-Y^2/2)$ when $Y \rightarrow \infty$. Next, we make the transformation $\Phi(Y) = \phi(Y) \exp(-Y^2/4)$ and arrive at the parabolic cylinder equation with index $2\beta + 1$ [38]

$$\phi'' + \left(2\beta + \frac{3}{2} - \frac{Y^2}{4}\right)\phi = 0. \quad (18)$$

This equation has two independent solutions: $D_{2\beta+1}(Y)$ and $D_{2\beta+1}(-Y)$ where D_ν is the parabolic cylinder function of index ν . Since the density vanishes, $\Phi \rightarrow 0$ as $Y \rightarrow \infty$, we choose the former solution, and therefore, $\Phi(Y) = D_{2\beta+1}(Y) \exp(-Y^2/4)$. The boundary condition $\Phi(A) = 0$ “selects” the persistence exponent β as the smallest root of the transcendental equation

$$D_{2\beta+1}(A) = 0, \quad (19)$$

in agreement with the announced result (3).

In terms of the original variables x and r , the joint density $P(x, r, t)$ has the asymptotic behavior (Fig. 2)

$$P(x, r, t) \sim t^{-1-\beta} D_{2\beta+1}\left(\frac{2r-x}{\sqrt{t}}\right) \exp\left[-\frac{(2r-x)^2}{4t}\right].$$

This distribution holds for $r > A\sqrt{t}$ and $-\infty < x < r$. We obtained the density $P(x, r, t)$ up to a prefactor that cannot be determined using scaling analysis alone. Finally, the asymptotic behavior $D_\nu(z) \sim z^\nu \exp(-z^2/4)$

shows that the density has a Gaussian tail $P(x, r, t) \sim \exp[-(2r - x)^2/(4t)]$.

It is straightforward to compute various moments of the joint distribution. In the long-time limit, these moments are directly related to moments of the scaling function $\Phi(Y)$, and it is convenient to use the adjusted moments $m_n = \int_A^\infty dY (Y^n - A^n) \Phi(Y)$. Remarkably, the average position of superior walks $\langle x \rangle_{\text{sup}}$ coincides with the average record, $\langle x \rangle_{\text{sup}} \simeq A\sqrt{t}$. As expected, the average record of superior walks grows faster, $\langle r \rangle_{\text{sup}} \simeq C\sqrt{t}$ with $C = m_2/(2m_1)$ or $C = 1.478591$. Finally, the position and the record are correlated random variables, $\langle xr \rangle_{\text{sup}} \neq \langle x \rangle_{\text{sup}} \langle r \rangle_{\text{sup}}$, as follows from $\langle xr \rangle_{\text{sup}} \simeq ct$ with $c = A^2/2 + m_3/(6m_1)$.

IV. INFERIOR WALKS

Inferior walks are simpler to analyze because they can be defined in terms of position alone: A walk is inferior if and only if $x(\tau) \leq a(\tau)$ for all $\tau = 0, 1, 2, \dots, t$. Indeed, if the position exceeds the average record, the record necessarily crosses the average. Conversely, if the position never exceeds the average record, then the record remains below average. Hence, inferior walks map onto diffusion in the presence of a receding trap with location that grows as square root of time, a problem that was solved in Ref. [39].

Since it is not necessary to keep track of the record, we study the distribution of position. Let $P(x, t)$ be the density of inferior walks with position x at time t . The probability $I(t)$ that a walk remains inferior after t steps is the integral of the density, $I(t) = \int_{-\infty}^{a(t)} dx P(x, t)$. The density of inferior walks obeys the diffusion equation

$$\frac{\partial P(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 P(x, t)}{\partial x^2}, \quad (20)$$

in the domain $-\infty < x < a(t)$, and is subject to the boundary condition $P(a, t) = 0$. We anticipate the scaling behavior

$$P(x, t) \sim t^{-\alpha-1/2} \Psi\left(\frac{x}{\sqrt{t}}\right), \quad (21)$$

and impose the boundary condition $\Psi(A) = 0$. The prefactor in (21) reflects the algebraic decay $I(t) \sim t^{-\alpha}$.

Substituting the scaling form (21) into the diffusion equation (20), we find that the scaling function obeys

$$\Psi'' + X\Psi' + (2\alpha + 1)\Psi = 0. \quad (22)$$

Here, prime denotes differentiation with respect to the scaling variable $X = x/\sqrt{t}$. With the transformation $\Psi(X) = \psi(X) \exp(-X^2/4)$, the function $\psi(X)$ obeys the parabolic cylinder equation

$$\psi'' + \left(2\alpha + \frac{1}{2} - \frac{X^2}{4}\right) \psi = 0. \quad (23)$$

This equation has two linearly independent solutions: $D_{2\alpha}(X)$ and $D_{2\alpha}(-X)$. The density should vanish as $X \rightarrow -\infty$ and this requirement gives $\psi(X) = D_{2\alpha}(-X)$. The boundary condition $\Psi(A) = 0$ leads to the transcendental equation stated in (3),

$$D_{2\alpha}(-A) = 0. \quad (24)$$

In terms of the original variables, the density of inferior walks has the asymptotic behavior

$$P(x, t) \sim t^{-\alpha-1/2} D_{2\alpha}\left(-\frac{x}{\sqrt{t}}\right) \exp\left[-\frac{x^2}{4t}\right]. \quad (25)$$

In particular, the density has a Gaussian tail $P(x, t) \sim \exp(-x^2/2t)$ as $x \rightarrow -\infty$.

Thus far, we considered only the maximal position, but one can also consider the maximal and minimal positions simultaneously. When $|x(\tau)| \leq a(\tau)$ for all $\tau = 0, 1, 2, \dots, t$, the minimal position and the maximal position are both inferior with respect to the average record. The density of such *meek* random walks satisfies the diffusion equation (20) in the growing interval $[-a(t), a(t)]$. We seek a scaling solution $P(x, t) \sim t^{-\gamma-1/2} \Psi(X)$ and arrive at the same ordinary differential equations (22)–(23) with a new persistence exponent γ replacing α . The density is symmetric, $\psi(X) = \psi(-X)$, and thus, $\psi(X) = D_{2\gamma}(X) + D_{2\gamma}(-X)$. The boundary condition $\Psi(A) = \Psi(-A) = 0$ leads to the transcendental equation

$$D_{2\gamma}(A) + D_{2\gamma}(-A) = 0 \quad (26)$$

that specifies the persistence exponent $\gamma = 1.698282\dots$. The probability that a walk is meek, $\int_{-a}^a dx P(x, t) \sim t^{-\gamma}$, is therefore much smaller than the probability that the walk is either superior or inferior.

V. SIMULATIONS AND EXTENSIONS

Figure 3 shows results of Monte Carlo simulations for the fractions S and I of superior and inferior walks. These results are in excellent agreement with the theoretical predictions. In the computations, we first calculated the average record $a(t)$ by considering M independent realizations of a discrete-time random walk with t steps. We then measured the fraction of all of realizations in which the maximal position of the walk is always larger or always smaller than the average record, $r(\tau) \geq a(\tau)$ or $r(\tau) \leq a(\tau)$ for all $\tau \leq t$.

To verify that the asymptotic behavior (2)–(3) is robust, we considered two distributions of jump length: (i) fixed step size: $\Delta = 1$ or $\Delta = -1$ with equal probabilities, and (ii) a variable step size, chosen with uniform probability in the domain $-1 \leq \Delta \leq 1$. The results shown in figure 3 are for the latter case and were obtained using $M = 10^6$ independent realizations.

Our analysis compared the record r with the average record $a \simeq A\sqrt{t}$. We can compare the record with

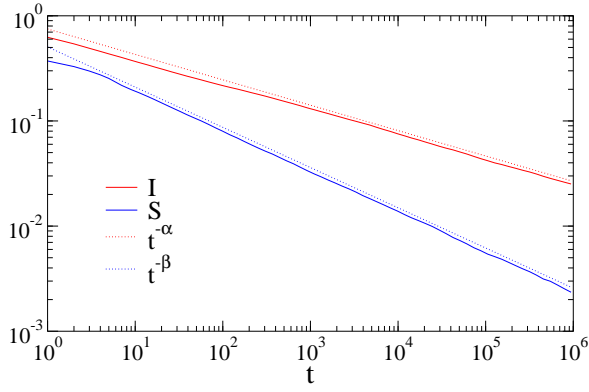


FIG. 3: The fractions S and I of superior and inferior walks versus time t . The simulation results represents 10^6 independent Monte Carlo runs. Also shown for reference are the theoretical results (2)–(3).

other length scales that grow as square root of time. We thus define a walk to be σ –superior [respectively σ –inferior] if $r(\tau) \geq \sigma \sqrt{\tau}$ [respectively $r(\tau) \leq \sigma \sqrt{\tau}$] for all $\tau = 0, 1, 2, \dots, t$. A straightforward generalization of the above analysis shows that the persistence exponents $\alpha(\sigma)$ and $\beta(\sigma)$ that govern the abundance of such walks are given by (Figure 4)

$$D_{2\alpha}(-\sigma) = 0 \quad \text{and} \quad D_{2\beta+1}(\sigma) = 0. \quad (27)$$

As expected, the persistence exponent $0 < \alpha < \infty$ increases monotonically with σ , while the exponent $0 < \beta < 1/2$ decreases monotonically. The maximal value $\alpha(0) = 1/2$ follows from the probability that a random walk remains in the negative half space, $S \sim t^{-1/2}$ [32]. The exponent α decays rapidly, $\alpha \simeq (\sigma/\sqrt{8\pi}) e^{-\sigma^2/2}$, while the exponent β diverges, $\beta \simeq \sigma^2/8$, in the limit $\sigma \rightarrow \infty$ (see [40]).

Figure 4 shows that both $\alpha < \beta$ and $\alpha > \beta$ are possible and that both persistence exponents vary continuously with σ . We note that for uncorrelated random variables, it was also found that $\alpha < \beta$ and $\alpha > \beta$ are both feasible, and that both exponents are continuous functions of some control parameter [35]. However, we do not believe that there is a deeper connection between these two sets of exponents or that it is possible to obtain persistence characteristics of records by mapping correlated random variables onto uncorrelated random variables.

Another useful by-product of our analysis is the joint distribution $P(x, r, t)$ of position and record for a one-dimensional random walk. If we do not impose any restriction on the record then $\beta = 0$ in Eqs. (16)–(17). The corresponding solution of (17) with $\beta = 0$ is $D_1(Y) \exp(-Y^2/4)$ and therefore,

$$P(x, r, t) \simeq \sqrt{\frac{2}{\pi t^3}} (2r - x) \exp \left[-\frac{(2r - x)^2}{2t} \right]. \quad (28)$$

Normalization of the probability distribution sets the numerical prefactor. By integrating (28) over position, one

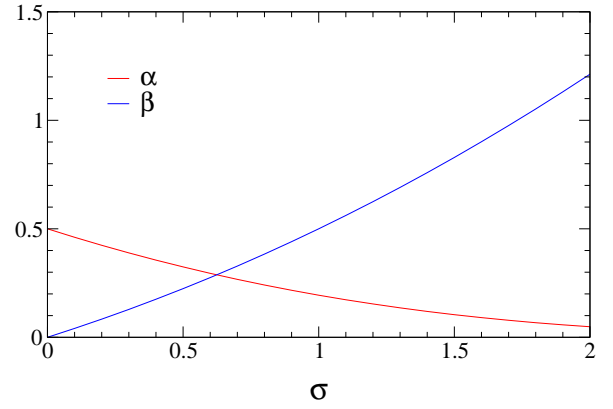


FIG. 4: The exponents α and β characterizing σ –inferior and σ –superior random walks.

recovers the record distribution (6). The joint distribution (28) has been originally discovered by Lévy, see [21, 22].

The joint distribution (28) shows that record and position are correlated variables. In particular, $\langle xr \rangle \simeq t/2$ whereas $\langle x \rangle = 0$ and $\langle r \rangle \simeq A\sqrt{t}$. Higher-order moments also reflect that x and r are correlated, and for example, $\langle x^2 r^2 \rangle \simeq 2t^2$ whereas $\langle x^2 \rangle = \langle r^2 \rangle = t$.

VI. CONCLUSIONS

In conclusion, we used the average record to characterize the motion of a Brownian particle in one dimension. A random walk is said to be superior if its maximal position is always above average and similarly, it is said to be inferior if the maximal position is always below average. We find that the probability that a walk is superior or inferior decays algebraically with time. This power-law decay is characterized by nontrivial persistence exponents.

For inferior walks, it suffices to keep track of the position of the walk alone and consequently, the problem reduces to diffusion in the presence of a properly-chosen moving trap. For superior walks, it is necessary to keep track of both record and position, and consequently, the problem involves diffusion in two-dimensional space. This random process consists of diffusion in the position coordinate and directional motion when the record and the position are equal. Nevertheless, a linear transformation of variables reduces the problem to one-dimensional diffusion in the presence of a moving trap as far as the asymptotic behavior is concerned (Figure 1).

Our results address the leading asymptotic behavior. However, different implementations of a random walk are not entirely equivalent. For instance, there are corrections to the leading behavior of the average (5), and $a(t)/\sqrt{\langle x^2(t) \rangle} = A + Ct^{-1/2} + \mathcal{O}(t^{-1})$. The constant C depends on the distribution of jump lengths and its derivation requires rather intricate analysis [41–43]. It will be interesting to understand how corrections to the

leading asymptotic behavior (2)–(3) depend on the distribution of jump lengths.

Finally, we mention that records have been extensively used to analyze empirical data such as earthquake inter-event times [44, 45] and temperature readings [46, 47]. Comparing a sequence of records with a baseline such as the average provides a measure of performance. The fraction of superior and inferior record sequences has

shown to be a sensible tool for analyzing earthquake data [35, 36]. Persistence of records is also useful in finance where it is natural to compare an individual stock price with the stock index [8–10].

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